Retake Percolation Theory

1 February 2024, 18:15-20:15

- It is absolutely not allowed to use calculators, phones, computers, books, notes, the help of others or any other aids.
- Write the answer to each question on a separate sheet, **with your name and student number on each sheet**. This is worth 10 points (out of a total of 100).

Exercise 1 (20 pts).
State and prove the BK inequality.

Exercise 2 (a:5, b:5, c:10 pts).
For odd m the *majority function* $\mathrm{Maj}_m : \{\pm 1\}^m \to \{\pm 1\}$ is given by

$$\mathrm{Maj}_m(x_1, \dots, x_m) := \begin{cases} +1 & \text{if } x_1 + \dots + x_m > 0, \\ -1 & \text{otherwise.} \end{cases}$$

That is, the outcome is whichever of the values $-1, +1$ is more common.

Let $n := k \cdot \ell$ for $k, \ell \in \mathbb{N}$ both odd. Consider the function $f : \{\pm 1\}^n \to \{\pm 1\}$ that is given by

$$f(x_1, \dots, x_n) := \mathrm{Maj}_\ell \left(\mathrm{Maj}_k(x_1, \dots, x_k), \mathrm{Maj}_k(x_{k+1}, \dots, x_{2k}), \dots, \mathrm{Maj}_k(x_{k(\ell-1)+1}, \dots, x_n) \right).$$

That is, we first take the majorities of the first k bits, the second k bits, etc.; and then we take the majority of the outcomes of the groups.

a) Show that $\mathbb{E} f = 0$.

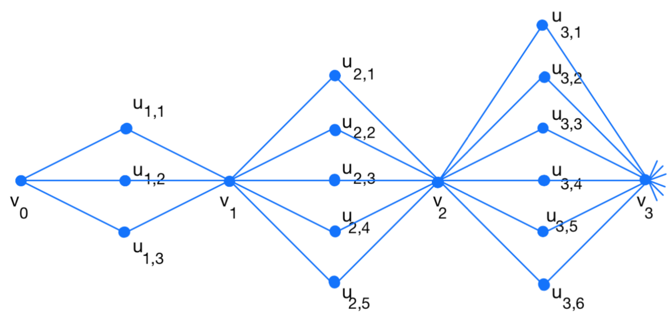
b) What is $\mathrm{Var} f$?

c) Show that, for $i = 1, \dots, n$:

$$\mathrm{Inf}_i(f) = \binom{k-1}{\frac{k-1}{2}} \cdot \binom{\ell-1}{\frac{\ell-1}{2}} \cdot 2^{2-(k+\ell)}.$$

(See next page)

Exercise 3 (a:6,b:6,c:6,d:6,e:6 pts)
In this exercise, we show that for every $0 < q < 1$ there exists an infinite planar graph $G = G(q)$ with critical probability $p_c(G) = q$. The construction of G is relatively straightforward. There are two kinds of vertices, $v_0, v_1, \ldots$, which we might draw consectutively on the x-axis. For each $i \geq 1$ there is a number n_i – to be specified later on – and there are vertices $u_{i,1}, \ldots, u_{i,n_i}$ that are each joined by an edge to v_{i-1} and to v_i. (There are no further edges or vertices in the graph.) See the picture below.



(Part of the graph G, assuming $n_1 = 3, n_2 = 5, n_3 = 6$.)

As usual, we declare each edge open with probability p, independently of all the other edges, and we say percolation occurs if there exists some infinite path consisting of open edges.
For $i \geq 1$, let A_i denote the event that there is an open path between v_{i-1} and v_i.

a) Explain why percolation occurs if and only if all but finitely many of the events A_i hold. Put differently:

$$\{\text{percolation}\} = \bigcup_{i=1}^{\infty} \bigcap_{j=i}^{\infty} A_j.$$

b) Explain why $\mathbb{P}(A_i) = 1 - (1 - p^2)^{n_i}$.

(One or two sentences should suffice.)

The time has come to reveal the choice of the values n_i. We will take

$$n_i := \left\lceil \frac{\ln i}{\ln\left(1/(1-q^2)\right)} \right\rceil.$$

c) Show that if $p > q$ then there exists a $\delta = \delta(p) > 0$ such that for all $i \geq 1$:

$$\mathbb{P}_p(A_i) \geq 1 - i^{-(1+\delta)},$$

while for $0 < p < q$ there exist $c = c(p), \delta = \delta(p)$ such that for all $i \geq 1$:

$$\mathbb{P}_p(A_i) \leq 1 - c \cdot i^{-(1-\delta)}.$$

d) Deduce from **c)** that if $p > q$ then $\mathbb{P}_p(\text{percolation}) = 1$.

(*Hint*: what can you say about $\lim_{i\to\infty} \mathbb{P}\left(\bigcap_{j\geq i} A_j\right)$?)

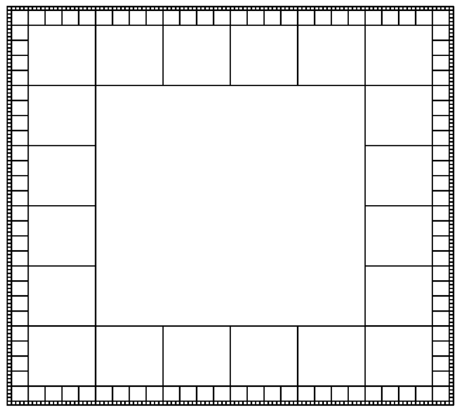
e) Deduce from **c)** that if $p < q$ then $\mathbb{P}_p(\text{percolation}) = 0$.

(*Hint*: You can start by arguing that it is enough to show $\mathbb{P}\left(\bigcap_{j\geq i} A_j\right) = 0$ for all i. Then you might use the independence of the events $A_1, A_2, \ldots$. As usual, you may use without proof that $1 - x \leq e^{-x}$.)

(See next page)

Exercise 4 (a:5,b:5,c:5,d:5 pts)
As explained during the lectures, site percolation on $\mathbb{Z}^2$ can alternatively be described as follows. We dissect the plane into axis-parallel squares of unit side-length. Squares are coloured black with probability p, independently of the colours of the other squares. Percolation now corresponds to the existence of an infinite, black, connected set of squares, where two individual squares are connected if they share a side (but not if they only share a corner).

We consider the analogous process, but on another dissection of the plane into squares.



(Part of a dissection of the plane into squares of different sizes.)

We start with the unit square, we place a layer of 20 squares of side length $1/4$ around it, we then place a layer of 84 squares of sidelength $1/16$ around that, and so on. (See the figure above for a depiction.)

Let N denote the number of infinite connected black clusters in the resulting percolation model (where squares are black with probability p independently, and two squares are only considered adjacent if they touch along a side of the smaller square.)

a) Show that if $p < 1/15$ then $\mathbb{P}_p(N = 0) = 1$.

(*Hint*: What is the maximum over all squares S in the disscetion of the number of squares adjacent to S?)

b) Show that if $p > 1/4$ then $\mathbb{P}_p(N \geq 1) = 1$.

(*Hint*: Consider "upwards black paths", where in each step we go to a (black) square adjacent and above the current square. Here and in the sequel you may use any relevant facts about Galton-Watson processes without proof, provided that you state them clearly and they have been covered in the course.)

c) Show that if $p > 18/19$ then $\mathbb{P}_p(N = 1) = 1$.

(*Hint*: If two distinct infinite black clusters exist, what can you say about the set of white squares?)

d) Show that if $1/4 < p < 3/4$ then $\mathbb{P}_p(N = \infty) = 1$.

(The end)